Fractional residues

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Abstract

Invariants of generalized tensor fields on a line are classified using special polynomials $P_{mk}^{(-1/\lambda)}$ introduced here for this purpose. For the case of positive characteristic, a new invariant of formal power series, *a width*, is defined. Some applications to the geometric quantization of a line and conformal quantum field theory are discussed as well.

1 Introduction

Differential forms $\phi(t)dt$ on a line have a well-known invariant,

$$\operatorname{Res}_0 \phi(t)dt = a_{-1}, \tag{1}$$

where

$$\phi(t) = \sum_{i=\operatorname{ord}\phi(t)}^{\infty} a_i t^i. \tag{2}$$

For quadratic differential forms $\psi(t)(dt)^2$, one can construct an invariant by a composition of an invariant mapping

$$\psi(t)(dt)^2 \mapsto \sqrt{\psi(t)}dt \tag{3}$$

and a residue (1).

Why do we need the invariants of quadratic differential forms? One of the reasons is that the pairing

$$(\psi(t)(dt)^2, \alpha(t)\frac{d}{dt}) = \mathrm{Res}_0\, \psi(t)\alpha(t)dt \tag{4}$$

identifies the space of the quadratic differentials with a dual space to the Lie algebra of vector fields on a line. Kirillov's orbit method [2, 4, 5] associates the orbits of the group of authomorphisms of a line in that space with irreducible unitary representations of this group. Thus, for geometric quantization of a line, we need to describe the orbits and the invariants of quadratic differentials.

The first calculations for that case were done by the founder of the orbit method, Alexandre Kirillov, in [3]. I was his student at that time, and I found the explicit formulas for the invariants, some of them were announced in [3] with an indication of my priority. These results are presented in section 2. Most of them are based on the studying of special polynomials P_{mk} parametrizing the orbits of the co-adjoint representation of the group of authomorphisms of a line.

More generally, the composition of the invariant mapping

$$\psi(t)(dt)^{-\lambda} \mapsto (\psi(t))^{-1/\lambda}dt \tag{5}$$

and a residue (1) defines an invariant of generalized differential forms $\psi(t)(dt)^{-\lambda}$. This nontrivial invariant allows us to describe the orbits of the group of authomorphisms in the space of generalized differential forms, utilizing special polynomials $P_{mk}^{(-1/\lambda)}$ which for $\lambda = -2$ coinside with polynomials P_{mk} introduced in section 2. The orbits and invariants of generalized differential forms are described in section 3.

Sections 2 and 3 deal with an arbitrary field of characteristic 0. Almost without changes, the results can be transfered to the restricted case of a positive characteristic $p > 0$. The results related to the geometric quantization of a line in the restricted case for $p > 0$, corresponding to section 2, are described briefly in section 4.

For the general, not restricted, case of a field f of a positive characteristic $p > 0$, the situation is much more complicated. In that case even functions have a lot of additional invariants. If f is not a perfect field, there are formal power series which don't have a polynomial normal form, see (54). However, all the orbits are closed, and the space of orbits can be metrized by a complete metrics. Section 5 presents these results. Also, at the end of section 5 I define a new invariant (67) of formal vector fields for $p > 0$, I called it a *width*.

Special polynomials $P_{mk}^{(-1/\lambda)}$ describing the orbits and invariants of formal tensor fields on a line, naturally appear in some other fields of mathematics as well. An application of them to a particular problem from a quantum field theory is discussed in section 6.

2 Polynomials P_{mk}

Let f be an arbitrary field of characteristic 0. Denote W_1 the Lie algebra of f-derivations of $f[[t]]$, the (associative) algebra of the formal power series in one variable. In other words, elements of W_1 are formal vector fields on a line $\mathbb{A}^1(f) = ft$, i. e. expressions

$$a = \sum_{i=-1}^{\infty} a_i l_i, \tag{6}$$

with $a_i \in f$ and $l_i = t^{i+1}\frac{\partial}{\partial t}$, with generators l_i satisfying

$$[l_i, l_j] = (j - i)l_{i+j}. \tag{7}$$

W_1 has a natural decreasing filtration

$$W_1 = L_{-1} \supset L_0 \supset L_1 \supset L_2 \supset \dots \tag{8}$$

where L_n is the Lie subalgebra of W_1, consisting of elements (6) with $a_i = 0$ if $i < n$. Since $[L_m, L_n] \subseteq L_{m+n}$, Lie algebra L_n is an ideal of L_m for all m such that $0 \le m < n$; and we can define Lie algebras $L_{mn} = L_m/L_n$ for $0 \le m < n$.

It follows directly from the definition, that L_{mn} is a Lie f-algebra of dimension $n - m$ with a basis $(l_i + L_n)_{m \le i < n}$ satisfying

$$[l_i + L_n, l_j + L_n] = \begin{cases} (j-i)(l_{i+j} + L_n) & \text{for } i + j < n, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Below we'll write l_i instead of $l_i + L_n$, where it won't cause an ambiguity, meaning that the brackets $[,]$ in L_{mn} satisfy (7) with $l_{i+j} = 0$ for $i + j \ge n$.

Filtration (8) defines a filtration

$$L_{mn} \supset L_{m+1,n} \supset \cdots \supset L_{nn} = 0 \tag{10}$$

with

$$[L_{in}, L_{jn}] = \begin{cases} L_{i+j,n} & \text{for } i \neq j, i+j < n, \\ L_{2i+1,n} & \text{for } i = j, 2i < n, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

according to (9). Thus L_{mn} are solvable Lie algebras, nilpotent for $m > 0$ and commutative for $2m + 1 \geq n$.

Denote U_{mn} the universal enveloping algebra of L_{mn}. (10) implies

$$U_{mn} \supset U_{m+1,n} \supset \cdots \supset U_{nn} = f \tag{12}$$

with $U_{in} = f[l_i, l_{i+1}, \ldots, l_{n-1}]$ for $(n-1)/2 \leq i \leq n-1$ since L_{in} is commutative in these cases.

Theorem 1. *Let $m \geq 0$. If $n \leq 2m+2$, $U_{m+1,n+1}$ is commutative. If $n \geq 2m+2$, the center of $U_{m+1,n+1}$ is $f[l_n, l_{n-1}, \ldots, l_{n-m}]$ for odd n and $f[l_n, l_{n-1}, \ldots, l_{n-m}, P_{mk}(l_n, l_{n-1}, \ldots, l_{n/2})]$ for even n, where $k = (n/2) - m$ and polynomials P_{mk} can be defined as the coefficients of a generating function*

$$\sum_{k=1}^{\infty} P_{mk}(x_0, x_1, \ldots, x_{m+k}) z^k = \frac{\sqrt{\sum_{k=0}^{\infty} x_k x_0^{k-1} z^k} - \sqrt{\sum_{k=0}^{m} x_k x_0^{k-1} z^k}}{x_0^m z^m} \tag{13}$$

A canonical projection $L_m \to L_{mn}$ induces a canonical inclusion of the spaces of f-linear forms, $L_{mn}^* \to L_m^*$, and one has an infinite flag

$$0 = L_{mm}^* \subset L_{m,m+1}^* \subset L_{m,m+2}^* \subset \cdots \subset L_m^* \tag{14}$$

with $L_m^* = \bigcup_k L_{m,m+k}^*$.

Denote $G_0 = \mathrm{Gal}(f((t))/f)$, the group of the automorphisms of $f((t))$, the field of power series, leaving the constants stable. Elements $g \in G_0$ can be uniquely determined by their values $g(t) = ct + o(t)$ with $c \in f^*$ i. e. $c \neq 0$.

For a positive integer n, denote G_n a subgroup of G_0 consisting of elements g satisfying $g(t) = t + o(t^n)$. For $m \leq n$, G_n is a normal subgroup of G_m. Denote $G_{mn} = G_m/G_n$.

For $m > 0$, there is a standard isomorphism between G_{mn} and an (algebraic) adjoint group of a nilpotent Lie algebra L_{mn}. The standard action of G_{mn} on L_{mn}^* coincides with the co-adjoint representation. The group G_m also may be considered as an adjoint group of a pronilpotent Lie algebra L_m for $m > 0$, and the standard action of G_m on L_m^* may be called a co-adjoint representation.

Theorem 2. *Let $m \geq 0$. If $n \leq 2m+2$, all the orbits of a co-adjoint representation of $G_{m+1,n+1}$ in $L_{m+1,n+1}$ are points, and every point is an orbit; all orbits are in a general position. If $n > 2m+2$, the orbits in a general position of a co-adjoint representation $G_{m+1,n+1}$ in $L_{m+1,n+1}$, can be parametrized by $m+1$ numbers $c_0 \in f^*$, $c_1, \ldots, c_m \in f$ for odd n: they are affine planes of dimension $n-2m-1$ defined by equations $l_n = c_0, l_{n-1} = c_1, \ldots, l_{n-m} = c_m$; or by $m+2$ numbers $c_0 \in f^*$, $c_1, \ldots, c_{m+1} \in f$ for even n, in which case they are affine varieties of dimension $n-2m-2$ defined by equations $l_n = c_0, l_{n-1} = c_1, \ldots, l_{n-m} = c_m, P_{mk}(l_n, l_{n-1}, \ldots, l_{n/2}) = c_{m+1}$ with $k = (n/2) - m$ and P_{mk} defined by (13). Each orbit in a general position of a co-adjoint representation of $G_{m+1,n+1}$ in $L^*_{m+1,n+1}$, is an orbit of a co-adjoint representation of $G_{m+1,i+1}$ in $L^*_{m+1,i+1}$ for all $i \geq n$, as well as of G_{m+1} in L^*_{m+1}. Each orbit of a co-adjoint representation of $G_{m+1,i+1}$ in $L^*_{m+1,i+1}$, is an orbit in a general position of a co-adjoint representation of $G_{m+1,n+1}$ in $L^*_{m+1,n+1}$ for some n such that $\min(2m+2, i) \leq n \leq i$. Each orbit of a co-adjoint representation of G_{m+1} in L^*_{m+1} is an orbit in a general position of a co-adjoint representation of $G_{m+1,n+1}$ in $L^*_{m+1,n+1}$ for some $n \geq 2m+2$.*

Let us study polynomials P_{mk} mentioned in Theorems 1 and 2 in more details.

Theorem 3. *A polynomial P_{mk} is homogeneous of degree k and equalized of weight $m+k$.*

$$P_{mk}(x_0, \ldots, x_{m+k}) = \sum_{\substack{\pi \vdash m+k \\ \pi_1 > m}} \binom{1/2}{p_1, \ldots, p_{m+k}} x_0^{p_0} x_1^{p_1} \ldots x_{m+k}^{p_{m+k}}, \tag{15}$$

where $\pi = (1^{p_1} 2^{p_2} \ldots)$ is supposed to be a partition of $m+k$ with the largest part $\pi_1 > m$; $p_0 = k - \ell(\pi)$ where $\ell(\pi)$ denotes the length of a partition π. The least common multiple of the denominators of the coefficients of P_{mk} equals $2^{2k-s(k)}$ for $m=0$ or $2^{2k-s(k-1)-1}$ for $m>0$, where $s(k)$ is the sum of digits of the binary expression of k. Also,

$$P_{mk}(x_0, \ldots, x_{m+k}) = \frac{(-1)^{k-1}}{(2k-2)!! \cdot 2} \int_0^x \det \binom{dx}{A}, \tag{16}$$

where $x = (x_{m+1}, \ldots, x_{m+k})$, $dx = (dx_{m+1}, \ldots, dx_{m+k})$ and

$$A = \begin{pmatrix} (2k-2)x_0 & (2k-3)x_1 & \ldots & kx_{k-2} & (k-1)x_{k-1} \\ 0 & (2k-4)x_0 & \ldots & (k-1)x_{k-3} & (k-2)x_{k-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 2x_0 & x_1 \end{pmatrix} \tag{17}$$

is a $(k-1) \times k$ matrix; $\binom{dx}{A}$ denotes $k \times k$ matrix obtained from A by adding a first row dx. Also,

$$P_{0k}(x_0, \ldots, x_k) = \frac{(-1)^{k-1}}{(2k)!!} \det \binom{x'}{A}, \tag{18}$$

where $x' = (x_1, 2x_2, \dots, kx_k)$ *and* A *as in* (17). *If* $m \geq k-1$, *then*

$$P_{mk}(x_0, \dots, x_{m+k}) = \frac{(-1)^{k-1}}{(2k-2)!! \cdot 2} \det \begin{pmatrix} x \\ A \end{pmatrix} \tag{19}$$

with $x = (x_{m+1}, \dots, x_{m+k})$ *and matrix* A *defined above in* (17). *One has* $P_{m1} = \frac{1}{2}x_{m+1}$;
$P_{02} = \frac{1}{8}(4x_0x_2 - x_1^2)$ *and* $P_{m2} = \frac{1}{4}(2x_0x_{m+2} - x_1x_{m+1})$ *for* $m \geq 1$;
$P_{03} = \frac{1}{16}(8x_0^2x_3 - 4x_0x_1x_2 + x_1^3)$, $\quad P_{13} = \frac{1}{16}(8x_0^2x_4 - 2x_0(2x_1x_3 + x_2^2) + 3x_1^2x_2)$,
$P_{m3} = \frac{1}{16}(8x_0^2x_{m+3} - 4x_0(x_1x_{m+2} + x_2x_{m+1}) + 3x_1^2x_{m+1})$ *for* $m \geq 2$;
$P_{04} = \frac{1}{128}(64x_0^3x_4 - 16x_0^2(2x_1x_3 + x_2^2) + 24x_0x_1^2x_2 - 5x_1^4)$,
$P_{14} = \frac{1}{32}(16x_0^3x_5 - 8x_0^2(x_1x_4 + x_2x_3) + 6x_0(x_1^2x_3 + x_1x_2^2) - 5x_1^3x_2)$,
$P_{24} = \frac{1}{32}(16x_0^3x_6 - 4x_0^2(2x_1x_5 + 2x_2x_4 + x_3^2) + 6x_0(x_1^2x_4 + 2x_1x_2x_3) - 5x_1^3x_3)$,
$P_{m4} = \frac{1}{32}(16x_0^3x_{m+4} - 8x_0^2(x_1x_{m+3} + x_2x_{m+2} + x_3x_{m+1}) + 6x_0(x_1^2x_{m+2} + 2x_1x_2x_{m+1}) - 5x_1^3x_{m+1})$
for $m \geq 3$. *Also,*

$$\begin{aligned} P_{0k}(1, 1, \dots, 1) &= \frac{(2k-1)!!}{(2k)!!} \\ P_{mk}(1, 1, \dots, 1) &= \frac{(2k-3)!!}{(2k-2)!! \cdot 2} \quad \text{for } m \geq k-1. \end{aligned} \tag{20}$$

If $m = 0$, *the sum* (15) *has* $p(k)$, *the number of partitions of* k, *nonzero items. If* $m \geq k-1$, *the sum* (15) *has* $p(0) + p(1) + \dots + p(k-1)$ *nonzero items.*

Referring to $x_0, x_1, \dots, x_m$ as constants, one obtains from (16),

$$dP_{mk} = \frac{\partial P_{mk}}{\partial x_{m+1}} dx_{m+1} + \dots + \frac{\partial P_{mk}}{\partial x_{m+k}} dx_{m+k} = \det \begin{pmatrix} dx \\ A \end{pmatrix}. \tag{21}$$

Expanding the determinant along the first row, we get determinant formulas for partial derivatives:

Corollary 1. *For an integer* i *so that* $1 \leq i \leq m$,

$$\frac{\partial P_{mk}}{\partial x_{m+i}} = (-1)^{i+1} \det A_i, \tag{22}$$

where A_i *is a matrix obtained from* A *by deleting* i*-th column.*

Lemma 1. *Let* r *be an arbitrary commutative ring,* $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$ *some derivations of* r, $P \in r$ *and*

$$dP \stackrel{\text{def}}{=} \frac{\partial P}{\partial x_1} dx_1 + \dots + \frac{\partial P}{\partial x_k} dx_k = \det \begin{pmatrix} dx_1 & \dots & dx_k \\ a_{21} & \dots & a_{2k} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} \tag{23}$$

with $a_{ij} \in r$. Then P satisfies the following system of partial differential equations:

$$\begin{cases} a_{21}\frac{\partial P}{\partial x_1} + \cdots + a_{2k}\frac{\partial P}{\partial x_k} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{k1}\frac{\partial P}{\partial x_1} + \cdots + a_{kk}\frac{\partial P}{\partial x_k} = 0. \end{cases} \tag{24}$$

Proof. Expanding the determinant

$$\det \begin{pmatrix} a_{i1} & \dots & a_{ik} \\ a_{21} & \dots & a_{2k} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{ik} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} = 0 \tag{25}$$

along the first row, we get the corresponding equation of the system (24). □

Proof of Theorem 1. By Gelfand's Lemma [1], for the standard representation of a Lie algebra having a basis $(x_0, \dots, x_N)$, on the polynomial algebra $f[x_0, \dots, x_N]$, one has

$$T(x_i)P = \sum_{j=0}^{N} [x_i, x_j] \frac{\partial P}{\partial x_j} \tag{26}$$

for all i from 0 to N and $P \in f[x_0, \dots, x_N]$. Thus the algebra of invariants of the given Lie algebra in $f[x_0, \dots, x_N]$, can be described as the algebra of solutions of a system of partial differential equations

$$\begin{cases} \sum_{j=0}^{N} [x_0, x_j] \frac{\partial P}{\partial x_j} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=0}^{N} [x_N, x_j] \frac{\partial P}{\partial x_j} = 0. \end{cases} \tag{27}$$

For a Lie algebra $L_{m+1,n+1}$ with odd n, the system (27) where $x_i = l_{n-i}$ and $N = n - m - 1$, has an upper triangular matrix with a non-zero main diagonal. Thus, by induction, invariants don't depend on l_i with $m < i < n - m$. If n is even, the matrix of the coefficients of system (27) for $L_{m+1,n+1}$ has a form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \tag{28}$$

where A is defined in (17), and C is an upper triangular square matrix with a non-zero main diagonal. By induction, the same as for the case of an odd n, one can deduct that invariants don't depend on l_i for $m < i < n/2$. Further, the invariants satisfy the system of partial differential equations with a matrix A. Now, utilizing Lemma 1 and formulas (16), (21), we get an additional invariant $P_{mk}(l_n, \dots, l_{n/2})$, with $k = (n/2) - m$.

It follows from the general theory of invariants of nilpotent Lie algebras [7], that the algebra of invariants of $L_{m+1,n+1}$ discussed above, for even n, is $f[l_n, l_{n-1}, \dots, l_{n-m}, P]$ with unknown polynomial P. Notice that our polynomial $P_{mk}(l_n, \dots, l_{n/2})$ with $k = (n/2) - m$, is not included in any algebras $f[l_n, l_{n-1}, \dots, l_{n-m}, P] \subseteq f[l_n, \dots, l_{n/2}]$ such that $P \notin f[l_n, \dots, l_{n-m}, P_{mk}(l_n, \dots, l_{n/2})]$, since

$$P_{mk}(l_n, \dots, l_{n/2}) = \frac{1}{2} l_n^{k-1} l_{n/2} + l_n Q_{mk}(l_n, \dots, l_{(n/2)+1}) + c_{mk} l_{n-1}^{k-1} l_{n-m-1} \quad (29)$$

for some polynomial Q_{mk} and nonzero constant c_{mk}, meaning that P_{mk} is a linear polynomial of $l_{n/2}$ with coprime coefficients.

$l_n, \dots, l_{n-m}$ are central elements of $L_{m+1,n+1}$. Connections between invariants and the center of $U_{m+1,n+1}$, the universal enveloping algebra, are well known now, and can be found in [1]. □

Another proof of Theorem 6, based on the studying of generating functions (13), will be given in the next section. Theorem 3 (except the determinant formula (18)) follows from a comparison between these two proofs of Theorem 1. The determinant formula can be obtained by differentiation of the corresponding generating function (13) and observing the conditions on coefficients; similar calculations can be found in [6] and [8]. Theorem 2 follows from Theorem 1 and the results of [7].

Some of results of this section were announced in [3].

3 Fractional residues

The same as in the previous section, let f be a field of characteristic 0. For $\lambda, \mu \in f$ denote $F_{\lambda\mu} = f[[t]]t^{\mu}(dt)^{-\lambda}$, a linear topological f-space with the topology induced from the standard topology of $f[[t]]$, the algebra of formal power series, assuming a discrete topology of f. Lie algebras L_m and groups G_m, with $m > 0$, naturally act on these spaces. The purpose of this section is to study the algebras $I^m_{\lambda\mu}$ of polynomial invariants of these actions.

Elements $e_k = t^{k+\mu}(dt)^{-\lambda}$, where $k = 0, 1, 2, \dots$, form a topological basis of $F_{\lambda,\mu}$. Denote $(x_k)_{0 \le k \in \mathbb{Z}}$ the dual basis of the topological f-space $F^*_{\lambda,\mu}$ of linear forms on $F_{\lambda,\mu}$.

Theorem 4. *Let m be a non-negative integer. If $\mu \neq (m+k+1)\lambda$ for any positive integers k, then $I^{m+1}_{\lambda\mu} = f[x_0, x_1, \dots, x_m]$. If $\lambda = \mu = 0$, then $I^{m+1}_{\lambda\mu} = f[x_0, x_1, \dots, x_m + 1]$. If $\lambda \neq$*

$0, \mu = (m+k+1)\lambda$ for a positive integer k, and $-1/\lambda \neq n$ for any positive integer $n < k$, then $I^{m+1}_{\lambda\mu} = f[x_0, x_1, \dots, x_m, P^{(-1/\lambda)}_{mk}(x_0, x_1, \dots, x_{m+k})]$ where polynomial $P^{(-1/\lambda)}_{mk}$ is defined by a generating function

$$\sum_{k=1}^{\infty} P^{(-1/\lambda)}_{mk}(x_0, x_1, \dots, x_{m+k}) z^k = \frac{(\sum_{i=0}^{\infty} x_i x_0^{i-1} z^i)^{-1/\lambda} - (\sum_{i=0}^{m} x_i x_0^{i-1} z^i)^{-1/\lambda}}{x_0^m z^m}. \quad (30)$$

If $-1/\lambda = n$ for a positive integer n and $\mu = (m+k+1)\lambda$ for a positive integer $k > n$, then $I^{m+1}_{\lambda\mu} = f[x_0, x_1, \dots, x_m, P^{(n)}_{mk}(x_0, x_1, \dots, x_{m+k})/x_0^{k-n}]$ where polynomial $P^{(n)}_{mk}$ is defined above.

Theorem 5. *Let m be a non-negative integer. If $\mu \neq (m+k+1)\lambda$ for any positive integers k, then the orbits in a general position of the standard representation of G_{m+1} in $F_{\lambda\mu}$ can be parametrized by $(m+1)$ numbers $c_0 \in f^*, c_1, \dots, c_m \in f$: they are affine planes of codimension $m+1$ given by equations $x_0 = c_0, x_1 = c_1, \dots, x_m = c_m$. If $\lambda \neq 0, \mu = (m+k+1)\lambda$ for a positive integer k, then the orbits in a general position of the standard representation of G_{m+1} in $F_{\lambda\mu}$ can be parametrized by $(m+2)$ numbers $c_0 \in f^*, c_1, \dots, c_m, c_{m+1} \in f$: they are affine varieties of codimension $m+2$ given by equations $x_0 = c_0, x_1 = c_1, \dots, x_m = c_m, P^{(-1/\lambda)}_{mk}(x_0, x_1, \dots, x_{m+k}) = c_{m+1}$ if $-1/\lambda \neq n$ for any positive integer $n < k$, or $x_0 = c_0, x_1 = c_1, \dots, x_m = c_m, P^{(-1/\lambda)}_{mk}(x_0, x_1, \dots, x_{m+k})/x_0^{k-n} = c_{m+1}$ if $-1/\lambda = n$ for a positive integer n, where polynomial $P^{(-1/\lambda)}_{mk}$ is defined by (30). Each orbit in a general position of the standard representation of G_{m+1} in $F_{\lambda\mu}$ is an orbit of the standard representation of G_{m+1} in $F_{\lambda,\mu-i}$ for each nonnegative integer i. Each orbit of the standard representation of G_{m+1} in $F_{\lambda\mu}$ is an orbit in a general position of the standard representation of G_{m+1} in $F_{\lambda,\mu+i}$ for a nonnegative integer i, with the only exception when $\lambda = 0$ and μ is a non-positive integer: then sets $c + \mathcal{O}$ are also orbits for any $c \in f^*$ and $\mathcal{O}$, an orbit in general position of the standard representation of G_{m+1} in F_{0i} for a positive integer i.*

Let us study polynomials $P^{(-1/\lambda)}_{mk}$ mentioned in Theorems 4 and 5 in more details.

Theorem 6. *A polynomial $P^{(-1/\lambda)}_{mk}$ is homogeneous of degree k and equalized of weight $m+k$.*

$$P^{(-1/\lambda)}_{mk}(x_0, \dots, x_{m+k}) = \sum_{\substack{\pi \vdash m+k \\ \pi_1 > m}} \binom{1/\lambda}{p_1, \dots, p_{m+k}} x_0^{p_0} x_1^{p_1} \dots x_{m+k}^{p_{m+k}}, \quad (31)$$

where $\pi = (1^{p_1} 2^{p_2} \dots)$ is supposed to be a partition of $m+k$ with the largest part $\pi_1 > m$; $p_0 = k - \ell(\pi)$ where $\ell(\pi)$ denotes the length of a partition π. Also,

$$P^{(-1/\lambda)}_{mk}(x_0, \dots, x_{m+k}) = \frac{1}{(k-1)!(-\lambda)^k} \int_0^x \det \begin{pmatrix} dx \\ A \end{pmatrix}, \quad (32)$$

where $x = (x_{m+1}, \dots, x_{m+k})$, $dx = (dx_{m+1}, \dots, dx_{m+k})$ *and*

$$A = \begin{pmatrix} (k-1)\lambda x_0 & ((k-1)\lambda+1)x_1 & \dots & ((k-1)\lambda+(k-2))x_{k-2} & (k-1)(\lambda+1)x_{k-1} \\ 0 & (k-2)\lambda x_0 & \dots & ((k-2)\lambda+(k-3))x_{k-3} & (k-2)(\lambda+1)x_{k-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda x_0 & (\lambda+1)x_1 \end{pmatrix} \tag{33}$$

is a $(k-1)\times k$ *matrix;* $\binom{dx}{A}$ *denotes* $k \times k$ *matrix obtained from* A *by adding a first row* dx*. Also,*

$$P_{0k}^{(-1/\lambda)}(x_0, \dots, x_k) = \frac{1}{k!(-\lambda)^k} \det \begin{pmatrix} x' \\ A \end{pmatrix}, \tag{34}$$

where $x' = (x_1, 2x_2, \dots, kx_k)$ *and* A *as in* (33)*. If* $m \geq k-1$*, then*

$$P_{mk}^{(-1/\lambda)}(x_0, \dots, x_{m+k}) = \frac{1}{(k-1)!(-\lambda)^k} \det \begin{pmatrix} x \\ A \end{pmatrix} \tag{35}$$

with $x = (x_{m+1}, \dots, x_{m+k})$ *and matrix* A *defined above in* (33)*. One has* $P_{m1}^{(-1/\lambda)} = -\frac{1}{\lambda}x_{m+1}$*;*
$P_{02}^{(-1/\lambda)} = -\frac{1}{\lambda}x_0x_2 + \frac{\lambda+1}{2\lambda^2}x_1^2$ *and* $P_{m2}(-1/\lambda) = -\frac{1}{\lambda}x_0x_{m+2} + \frac{\lambda+1}{\lambda^2}x_1x_{m+1}$ *for* $m \geq 1$*;*
$P_{03}^{(-1/\lambda)} = -\frac{1}{\lambda}x_0^2x_3 + \frac{\lambda+1}{\lambda^2}x_0x_1x_2 - \frac{(\lambda+1)(2\lambda+1)}{6\lambda^3}x_1^3$,
$P_{13}^{(-1/\lambda)} = -\frac{1}{\lambda}x_0^2x_4 + \frac{\lambda+1}{2\lambda^2}x_0(2x_1x_3 + x_2^2) - \frac{(\lambda+1)(2\lambda+1)}{2\lambda^3}x_1^2x_2$,
$P_{m3}^{(-1/\lambda)} = -\frac{1}{\lambda}x_0^2x_{m+3} + \frac{\lambda+1}{\lambda^2}x_0(x_1x_{m+2} + x_2x_{m+1}) - \frac{(\lambda+1)(2\lambda+1)}{2\lambda^3}x_1^2x_{m+1}$ *for* $m \geq 2$*;*
$P_{04}^{(-1/\lambda)} = -\frac{1}{\lambda}x_0^3x_4 + \frac{\lambda+1}{2\lambda^2}x_0^2(2x_1x_3 + x_2^2) - \frac{(\lambda+1)(2\lambda+1)}{2\lambda^3}x_0x_1^2x_2 + \frac{(\lambda+1)(2\lambda+1)(3\lambda+1)}{24\lambda^4}x_1^4$,
$P_{14}^{(-1/\lambda)} = -\frac{1}{\lambda}x_0^3x_5 + \frac{\lambda+1}{\lambda^2}x_0^2(x_1x_4 + x_2x_3) - \frac{(\lambda+1)(2\lambda+1)}{2\lambda^3}x_0(x_1^2x_3 + x_1x_2^2) + \frac{(\lambda+1)(2\lambda+1)(3\lambda+1)}{6\lambda^4}x_1^3x_2$,
$P_{24}^{(-1/\lambda)} = -\frac{1}{\lambda}x_0^3x_6 + \frac{\lambda+1}{2\lambda^2}x_0^2(2x_1x_5 + 2x_2x_4 + x_3^2) - \frac{(\lambda+1)(2\lambda+1)}{2\lambda^3}x_0(x_1^2x_4 + 2x_1x_2x_3) + \frac{(\lambda+1)(2\lambda+1)(3\lambda+1)}{6\lambda^4}x_1^3x_3$,
$P_{m4}^{(-1/\lambda)} = -\frac{1}{\lambda}x_0^3x_{m+4} + \frac{\lambda+1}{\lambda^2}x_0^2(x_1x_{m+3} + x_2x_{m+2} + x_3x_{m+1}) - \frac{(\lambda+1)(2\lambda+1)}{2\lambda^3}x_0(x_1^2x_{m+2} + 2x_1x_2x_{m+1}) + \frac{(\lambda+1)(2\lambda+1)(3\lambda+1)}{6\lambda^4}x_1^3x_{m+1}$ *for* $m \geq 3$*. Also,*

$$\begin{aligned} P_{0k}^{(-1/\lambda)}(1, 1, \dots, 1) &= (-1)^k \binom{1/\lambda}{k}, \\ P_{mk}^{(-1/\lambda)}(1, 1, \dots, 1) &= \frac{(-1)^k}{\lambda} \binom{1/\lambda}{k-1} \quad \textit{for } m \geq k-1. \end{aligned} \tag{36}$$

If $-1/\lambda \neq n$ *for any positive integer* $n < k$*, the sum* (31) *has* $p(k)$ *nonzero items for* $m = 0$*, or* $p(0) + p(1) + \dots + p(k-1)$ *nonzero items for* $m \geq k-1$*. If* $-1/\lambda = n$ *for a positive integer* $n < k$*, then the sum* (31) *has* $p_n(k)$*, the number of partitions of* k *with length* $\leq n$*, nonzero items for* $m = 0$*, or* $p_n(0) + p_n(1) + \dots + p_n(k-1)$ *nonzero items for* $m \geq k-1$*.*

Proof of Theorem 4. Analogously to the proof of Theorem 6, one can notice that for a representation T of a Lie f-algebra with a basis $(l_i)_{i \in J}$, in the algebra $f[x_0, x_1, \dots]$, one has the series of equalities

$$T(l_i)P = \sum_{j \in J} (l_i x_j) \frac{\partial P}{\partial x_j}. \tag{37}$$

Continuing as in the proof of Theorem 1, we get a proof of Theorem 4.

There is also another proof. Consider a residue $\mathrm{Res}_0 = x_k \in I_{-1,-k-1}$ where k is a nonnegative integer. For such a non-negative k, if $\mu = (k+1)\lambda$ and $\lambda \neq 0$, then one has an invariant polynomial mapping

$$\begin{aligned} P^{(-1/\lambda)} : F_{\lambda\mu}\big|_{x_0=1} &\longrightarrow F_{-1,-k-1} \\ ht^{\mu}(dt)^{-\lambda} \mapsto (ht^{\mu}(dt)^{-\lambda})^{-1/\lambda} &= h^{-1/\lambda}t^{-k-1}dt. \end{aligned} \tag{38}$$

Composing this invariant mapping with a standard residue Res_0, we obtain a polynomial invariant for every positive integer k,

$$\mathrm{Res}_0 \circ P^{(-1/\lambda)} : F_{\lambda\mu}\big|_{x_0=1} \longrightarrow f. \tag{39}$$

Noticing that x_0 is also a G_1-invariant, we can extend (39) first to a rational G_1-invariant

$$\mathrm{Res}_0 \circ P^{(-1/\lambda)} \circ \left(\frac{\cdot}{x_0}\right) : F_{\lambda\mu}\big|_{x_0\neq 0} \longrightarrow f \tag{40}$$

and then to a polynomial G_1-invariant

$$P_{0k}^{(-1/\lambda)} = \left(\cdot x_0^k\right) \circ \mathrm{Res}_0 \circ P^{(-1/\lambda)} \circ \left(\frac{\cdot}{x_0}\right) : F_{\lambda\mu} \longrightarrow f. \tag{41}$$

The rest of the proof can be done by utilizing the standard techniques from [7]. □

Noticing that bilinear transformations

$$P : F_{\lambda\mu} \times F_{\lambda'\mu'} \longrightarrow F_{\lambda+\lambda',\mu+\mu'}, (e_i, e_j) \mapsto e_{i+j} \tag{42}$$

are invariant, one obtains a bilinear invariant for $\lambda + \lambda' = -1$ when $\mu + \mu'$ is a negative integer:

$$\mathrm{Res}_0 \circ P : F_{\lambda\mu} \times F_{\lambda'\mu'} \longrightarrow f. \tag{43}$$

Thus,

$$F_{\lambda\mu}^* \simeq \left(\bigcup_i F_{-1-\lambda,i-\mu}\right) / F_{-1-\lambda,-\mu} \stackrel{\text{def}}{=} F_{-1-\lambda,-\mu}^-. \tag{44}$$

In particular,

$$L_m^* = F_{1,m+1} \simeq F_{-2,-1-m}^- \tag{45}$$

and

$$L^*_{mn} \simeq F_{-2,-1-n}/F_{-2,-1-m}. \tag{46}$$

That explains the identity

$$P_{mk} = P^{(1/2)}_{mk} \tag{47}$$

following from (13) and (30).

Caution. 'An orbit in a general position', here and in the previous chapter, doesn't mean the 'orbit of maximal dimension' or the 'orbit of minimal codimension'. For instance, the following two series of the orbits of the co-adjoint representation of $G_{1,5}$: given by equations

$$l_4 = c_0, \quad \frac{1}{2}l_4 l_2 - \frac{1}{8}l_3^2 = c_1 \tag{48}$$

with $c_0 \in f^*, c_1 \in f$, and by equations

$$l_4 = 0, \quad l_3 = c_0 \tag{49}$$

with $c_0 \in f^*$, both have the maximal dimension 2, but only (48) are orbits in a general position of the co-adjoint representation of $G_{1,5}$ in $L^*_{1,5}$.

4 Positive characteristic, a restricted case

Let f be an arbitrary field of a positive characteristic p. Denote W_1 the restricted Lie p-algebra of f-derivations of $f[t]/(t^p)$. Elements of W_1 can be written in a form

$$a = \sum_{i=-1}^{p-2} a_i l_i \tag{50}$$

with $a_i \in f$ and $l_i = t^{i+1}\frac{d}{dt}$. The basic elements l_i satisfy (7) meaning $l_{i+j} = 0$ for $i + j \geq p - 1$. Also $l_i^p = 0$ for $i \neq 0$, and $l_0^p = l_0$.

The same as in section 2, consider filtration

$$W_1 = L_{-1} \supset L_0 \supset \cdots \supset L_{p-2} \tag{51}$$

assuming that L_n is a Lie p-algebra consisting of expressions (50) with $a_i < 0$ for $i < n$. Define $L_{mn} = L_m/L_n$ for $0 \leq m \leq n \leq p-1$, supposing that $L_{m,p-1} = L_m$. Denote U_{mn} the restricted universal enveloping algebra of L_{mn}.

Theorem 7. *Let $m \geq 0$. If $n \leq 2m+2$, then $U_{m+1,n+1}$ is commutative. If $p-2 \geq n \geq 2m+2$, the center of $U_{m+1,n+1}$ is $f[l_n, l_{n-1}, \dots, l_{n-m}]/(l_n^p, l_{n-1}^p, \dots, l_{n-m}^p)$ for odd n, or $f[l_n, l_{n-1}, \dots, l_{n-m}, P_{mk}(l_n, l_{n-1}, \dots, l_{n/2})]/((l_n^p, \dots, l_{n/2}^p) \bigcap f[l_n, l_{n-1}, \dots, l_{n-m}, P_{mk}(l_n, l_{n-1}, \dots, l_{n/2})])$ for even n, where $k = (n/2) - m$, and P_{mk} defined by (13).*

Denote G_0 the group of automorphisms of $f[t]/(t^p)$, leaving the constants stable, and G_n with $1 \leq n \leq p-1$, the subgroup of G_0 of automprphisms $g(t) = t + o(t^n)$. Also denote $G_{mn} = G_m/G_n$ for $0 \leq m \leq n \leq p-1$.

Theorem 8. *Theorem 2 is true mutatis mutandis.*

The proofs of Theorems 7 and 8 can be obtained the same way as the proofs of Theorems 1 and 2, mutatis mutandis.

5 Formal singularities

Let f be an arbitrary field of characteristic $p \geq 0$. Denote $F = f((t))$ and $G_0 = \mathrm{Gal}(F/f)$. We suppose that F has a valuation, a filtration and a topology, as usual, assuming a discrete topology of f. One can check that all of the elements of G_0 are automatically continous automorphisms preserving the valuation, and are defined uniquely by their values $g(t) = ct + o(t)$ with $c \in f^*$, i. e. $c \neq 0$.

For a positive integer n, denote G_n the subgroup of G_0 of automorphisms g such that $g(t) = t + o(t^n)$. The filtration

$$G_0 \supset G_1 \supset G_2 \supset \dots \tag{52}$$

defines a topology ('given by a filtration') in every G_n.

Theorem 9. *Let n be a nonnegative integer. G_n-orbit of a formal power series $h \in F$, is open iff $h \in F \setminus f((t^p))$. The statement, '$G_n$-orbit of $h \in f((t^{p^m})) \setminus f((t^{p^{m+1}}))$ is open in the relative topology of a field $f((t^{p^m}))$' is true for all $h \in F \setminus f$, iff f is a perfect field. All the series $h \in F \setminus f((t^p))$ have a polynomial normal form, i. e. $G_n(h) \bigcap f[t^{-1}, t] \neq \emptyset$. All the series $h \in F$ have a polynomial normal form, iff f is a perfect field. Every G_n-orbit in F is closed. The canonical projection*

$$\beta : F \longrightarrow F/G_n, \quad h \mapsto G_n(h) \tag{53}$$

is closed iff either f is a finite field, or $p = 0$. The orbit space F/G_n can be metrized by a complete metrics.

Proof. Start with a counterexample to the existence of a polynomial normal form for the case of imperfect field. Let f be an imperfect field, $c \in f \setminus f^p$ and

$$h = \sum_{i=1}^{\infty} c^{p_i} t^{p^i} \tag{54}$$

where

$$p_i = 1 + p + p^2 + \cdots + p^i = \frac{p^{i-1} - 1}{p - 1}. \tag{55}$$

Since h satisfies

$$h - ch^p = t^p, \tag{56}$$

one has

$$g(h) - cg(h)^p = g(t)^p \tag{57}$$

for any $g \in G_n$. Suppose that $g(h)$ is a polynomial of degree d. Then $d \geq p$, because as we noticed, g saves the valuation, and $\operatorname{ord} h = p$. Calculating coefficients at t^{d^p} in (57), we obtain

$$c = -\left(\frac{g(t)_d}{g(h)_d}\right)^p \in f^p, \tag{58}$$

a contradiction. Thus, the series (54) doesn't have a polynomial normal form for any imperfect field f.

The next interesting fact, that all the orbits are closed, follows from a minimality principle, one of the formulations of Hilbert's theorem about bases of polynomial rings, and the following

Lemma 2. *Let K be an algebraically closed field containing f; $\mathbb{A}^m$ an affine K-space of a finite dimension m, and $\mathbb{A}^m(f)$ the set of f-rational points of $\mathbb{A}^m$. For each polynomial function $\Phi : \mathbb{A}^m \to f$ and for each f-linear subspace L of the f-linear space K, one can find an f-closed affine algebraic variety $S \subseteq \mathbb{A}^m$ such that $S(f) = \Phi^{-1}(L) \bigcap \mathbb{A}^m(f)$.*

In calculations involving the field F, the following lemma is extremely useful.

Lemma 3. *Let* $p > 0$, $k =^p \overline{\ldots k_i \ldots k_1 k_0} \in \mathbb{Z}_p$*, a* p*-adic integer,* $q_1 =^p \overline{\ldots q_{1i} \ldots q_{11} q_{10}}$*,* $q_2 = {}^p\overline{\ldots q_{2i} \ldots q_{21} q_{20}}, \dots \in \mathbb{Z}_{>0}$*, a finite sequence of nonnegative integers,* $q = q_1 + q_2 + \dots$ *. Then*

$$\binom{k}{q_1, q_2, \dots} \in \mathbb{Z}_p, \tag{59}$$

$$\binom{k}{q_1, q_2, \dots} \equiv \prod_{i=0}^{\infty} \binom{k_i}{q_{1i}, q_{2i}, \dots} \bmod p, \tag{60}$$

$$\binom{k}{q_1, q_2, \dots} \not\equiv 0 \bmod p \quad \textit{iff} \quad \forall i,\ k_i \geq q_{1i} + q_{2i} + \dots \ . \tag{61}$$

For (61)*, one has* $\forall i,\ \nu_p(q_i) \geq \nu_p(k)$ *where* ν_p *denotes the* p*-adic valuation.*

Proof. If $k \in \mathbb{Z}_{>0}$, a positive integer, then (59) is clear, (60) follows from the particular case of binomial coefficients, which is well-known, and (61) follows from (60). Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we can extend (59), (60) and (61) to $k \in \mathbb{Z}_p$ by continuity. □

By the way, we obtained all the formulas (59), (60) and (61) for negative integers k as well, just from the case of positive k and density of $\mathbb{Z}$ in $\mathbb{Z}_p$. For me, that is a very interesting p-adic trick.

The last sentence of Theorem 9 follows from

Lemma 4. *Suppose that a group* Γ *acts on a metric space* (X, d) *by isometries such that all the orbits are closed. Then*

(i) Function $D: (X/\Gamma)^2 \to \mathbb{R},\ (A, B) \mapsto \inf_{(a,b)\in A\times B} d(a, b)$ *is a metric on* X/Γ *defining the quotient topology.*

(ii) $D(A, B) = d(a, B)$ *for any* $a \in A,\ (A, B) \in (X/\Gamma)^2$*.*

(iii) A canonical injection in : $X/\Gamma \to H(X)$ *where* $(H(X), d_H)$ *is the space of non-empty closed subsets of* X *with Hausdorf's metrics*

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}, \tag{62}$$

is an isometry and $\mathrm{in}(X/\Gamma)$ *is a closed subset of* $H(X)$*.*

(iv) If (X, d) *is a complete metric space, then* $(X/\Gamma, D)$ *is a complete metric space as well.*

□

Theorem 9 shows that the situation for $p > 0$ is much more complicated than it is for $p = 0$ which was studied in section 3. In addition to invariants described previously, there are a lot of new invariants for $p > 0$. For instance, the following functions $F \to \mathbb{Z} \bigcup \{\infty\}$ are invariant:

$$\mathrm{ord}_0(h) = \mathrm{ord}(h - x_0(h)), \tag{63}$$

$$\mathrm{md}(h) = \max\{m \in \mathbb{Z} | h \in f((t^{p^m}))\}, \tag{64}$$

$$\mathrm{ord}_{\mathrm{md}}(h) = \max \mathrm{ord}\{h - a | a \in f((t^{p^{\mathrm{md}(h)+1}}))\}, \tag{65}$$

where

$$h = \sum_{i=\mathrm{ord}\, h}^{\infty} x_i(h) t^i \tag{66}$$

with $x_i(h) \in f$ and $x_{\mathrm{ord}\, h}(h) \neq 0$.

The next example is more interesting. Determine $w : F \to \mathbb{Z}$,

$$w(h) = \max \left\{ \left[\frac{\mathrm{ord}_{\mathrm{md}}(h) - m}{|m|_p^{-1} - |\,\mathrm{ord}_{\mathrm{md}}(h)|_p^{-1}} \right] \mid x_m(h) \neq 0, 0 \neq m < \mathrm{ord}_{\mathrm{md}}(h) \right\} \tag{67}$$

for $\mathrm{ord}_{\mathrm{md}}(h) > \mathrm{ord}_0(h)$, or $w(h) = 0$ otherwise. Here

$$|m|_p^{-1} \stackrel{\mathrm{def}}{=} \max \left\{ p^d \mid \frac{m}{p^d} \in \mathbb{Z}, d \in \mathbb{Z} \right\} \tag{68}$$

Letter w in (67) is the first letter of the word *width*.

Proposition 1. *w is G_0-invariant.*

6 An application to QFT

Polynomials $P_{mk}^{(-1/\lambda)}$ describing the orbits and invariants of formal tensor fields on a line, naturally appear in some other fields of mathematics as well. Here is just an example.

According to [9], denote

$$P_2 = u_2 - u_1^2, \tag{69}$$

$$P_{k+1} = \frac{1}{k+2} \left(\sum_{i=1}^{k} ((i+2)u_{i+1} - 2u_1 u_i) \frac{\partial P_k}{\partial u_i} - 2k u_1 P_k - \sum_{i=2}^{k-1} P_i P_{k+1-i} \right) \tag{70}$$

for $k > 2$.

Theorem 10. *For $k > 1$,*

$$P_k(u_1, u_2, \ldots, u_k) = \frac{1}{1-k} P_{0k}^{(1-k)}(1, u_1, u_2, \ldots, u_k). \tag{71}$$

Proof. As usual in analogous cases, after guessing the answer, the proof can be done by induction on k. Limited by the space and time, I omit superfluous details. □

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