

Symplectic tensor invariants, wave graphs and S-tris ^{*}

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Abstract

The spaces of invariants of tensor powers of the defining representation of $Sp(2n)$ are provided with the bases parametrized by symplectic wave graphs introduced here especially for this purpose. The proof utilizes a game similar to Tetris, named here S-tris. This work continues my previous work [16] on the tensor invariants of $SL(n)$, wave graphs and L-tris.

1 Introduction

Rumer, Teller and Weyl [21] parametrized a basis of the subspace of $SL(2)$ -invariants of $V^{\otimes m} = V \otimes \cdots \otimes V$ (m times), where V is the two-dimensional linear space with the standard action of $SL(2)$, by 1-regular outerplanar graphs, i. e. graphs with the vertices $1, 2, \dots, m$, edges of which can be drawn in the upper half-plane without intersections. They used slightly different graphs, drawn as a set of non-intersecting chords inside a disk, but after a conformal mapping of a disk onto the upper half plane and the indexing of the vertices in the increasing order, one gets the graphs described above.

This theory was developed and applied to the percolation theory by Temperley and Lieb [22], to the knots theory and invariants of 3-manifolds by Jones [7], Kauffman [8], Kauffman and Lins [9], Wenzl [24], Jaeger [6], Lickorish [11], Masbaum and Vogel [13] and others, to the quantum theory by

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Penrose [19] and Moussouris [17], to quantum groups and the quantum link theory by Reshetikhin and Turaev [20], Ohtsuki and Yamada [18], Carter, Flath and Saito [1] and others, to the theory of Lusztig's canonical bases [12] by Khovanov and Frenkel [2], Varchenko [23] and Frenkel, Varchenko and Kirillov, Jr. [3].

Furlan, Stanev and Todorov [4] have extended outerplanar $SL(2)$ invariants to the quantum algebra $U_q(sl(2))$ (for arbitrary spins). I became familiar with the description of the basis of the invariants of the tensor products of any irreducible representations of $SL(2)$ in the terms of the outerplanar graphs from Kuperberg's work [10]. In [14] I gave a new proof of a classical theorem of Rumer, Teller and Weyl [21] and its generalization for the case of arbitrary spins. In [15] I parametrized by outerplanar graphs the bases in the decompositions of any (repeated) tensor products of polynomial representations of $SL(2)$. Instead of the classical approach to the invariant theory using the straightening method, I used in [14, 15] the linear independence reason and the enumeration of the outerplanar graphs.

In recent work [16] I provided the spaces of invariants of tensor powers of the defining representations of $SL(n)$ with the bases parametrized by wave graphs introduced there especially for this purpose. The proof utilized a game similar to Tetris, named there L-tris, as well as the same linear independence reason as for the case $n = 2$ and the enumeration of wave graphs.

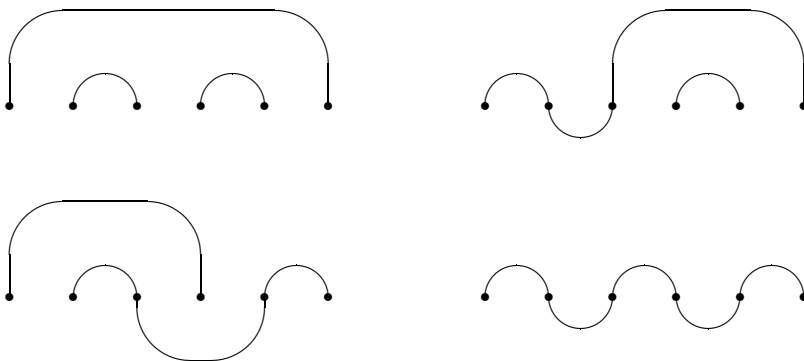
Here I give similar constructions for $Sp(2n)$, parametrizing the invariants of tensor powers of the defining representation of $Sp(2n)$ by symplectic wave graphs introduced here especially for this purpose. The proof utilizes a game similar to L-tris, named here S-tris, as well as the same linear independence reason and the enumeration of symplectic wave graphs.

I am preparing an article providing the space of invariants of tensor powers of the defining representations of orthogonal groups with the basis parametrized by (odd or even) orthogonal wave graphs introduced there especially for that purpose. Since we have a few different kinds of wave graphs, I propose to add to the name of wave graphs introduced in [16] the adjective 'linear', i. e. refer to them as linear wave graphs and use the term *wave graphs* for all of them: linear, symplectic and odd or even orthogonal wave graphs, as well as, exceptional wave graphs for the exceptional Lie groups.

A symplectic $2n$ -wave graph is a graph with the vertices $1, 2, \dots, m$, each connected component of which is a path of length ≥ 1 (i. e. it can't be a point), edges of which can be drawn in the book

with n pages, i. e. n copies of the upper half-plane, glued along \mathbb{R} , such that the first edge of each connected component, $\{i_1 i_2\}$, is drawn on the first page; each edge $\{i_j i_{j+1}\}$ consequent to the edge $\{i_{j-1} i_j\}$ drawn on k -th page, is drawn either on $(k+1)$ -th or $(k-1)$ -th page, if they exist, I mean that the edge consequent to the edge drawn on the first page, must be drawn on the second page and the edge consequent to the edge drawn on the n -th page, must be drawn on $(n-1)$ -th page; the last edge of the path, $\{i_l i_{l+1}\}$, supposed to be drawn on the first page; we suppose also that $i_1 < i_2 < \dots < i_{l+1}$ and edges of our symplectic wave graph don't intersect.

Symplectic 2-graphs are exactly 1-regular outerplanar graphs, or linear 2-wave graphs which is not surprising because $Sp(2) = SL(2)$. Here are 4 of a total number of 14 of symplectic 4-wave graphs with 6 vertices:



The corresponding invariants are

$$(\omega \otimes \omega \otimes \omega)^{(26543)}, \quad ((\omega \wedge \omega) \otimes \omega)^{(465)} \quad (1)$$

$$((\omega \wedge \omega) \otimes \omega)^{(1235)(46)}, \quad \omega \wedge (\omega \otimes \omega) \quad (2)$$

where

$$\omega = p_1 \wedge q_1 + p_2 \wedge q_2 \quad (3)$$

and for $\sigma \in S_6$, $t \in V^{\otimes 6}$ where V is the defining representation of $Sp(4)$, the tensor t^σ is the result of permutation σ applied to the components of t ; also, for 2-tensor α and $(m-2)$ -tensor β we define m -tensor

$$\alpha \wedge \beta = \sum_{1 \leq i < j \leq m} (-1)^{i+j-3} (\alpha \otimes \beta)^{\sigma_{ij}} \quad (4)$$

where $\sigma_{ij} \in S_m$ is the permutation mapping 1 to i , 2 to j and other elements to the vacant places in increasing order, i. e. $\sigma_{ij}(3) < \sigma_{ij}(4) < \dots < \sigma_{ij}(m)$.

For a symplectic wave graph G , denote t_G the analogous tensor products of the basic invariants corresponding to the connected components, see Definition 3.

Theorem 1. *Tensors t_G parametrized by all $2n$ -wave graphs with m vertices, form a basis in the space of $Sp(2n)$ -invariants in $V^{\otimes m}$, where V is the $2n$ -dimensional space of the defining representation of $Sp(2n)$.*

The proof uses a game similar to Tetris, named here S-tris, linear independence reason, explicit formulas for the invariants and the enumeration of symplectic wave graphs.

2 The main theorem

In this section we give all the necessary definitions and prove the main theorem.

Let f be a field of characteristic 0 and $Sp(2n)$ —the group of $2n \times 2n$ f -matrices acting on $2n$ -dimensional linear f -space V with basis $B_{2n} = (p_1, \dots, p_n, q_1, \dots, q_n)$ by the standard way, preserving the symplectic 2-form

$$\omega = p_1 \wedge q_1 + \dots + p_n \wedge q_n. \quad (5)$$

Recall some fundamental facts about the representations of $Sp(2n)$, see [5]. The word *representation* will mean below a polynomial finite dimensional linear representation over f . Every representation of $Sp(2n)$ is equivalent to a sum of irreducible representations. All classes of equivalence of the irreducible representations are parametrized by partitions of length $\leq n$. Denote P_n the set of partitions of length $\leq n$ and denote $\tilde{\rho}_\lambda$ the irreducible representation of $Sp(2n)$ corresponding to a partition $\lambda \in P_n$. Then $\tilde{\rho}_0$ is a trivial representation of dimension 1 and $\tilde{\rho}_1$ is the standard representation in V mentioned above. To describe the decomposition of tensor products of irreducible representations, we'll use Young diagrams.

The same as in [16], let us draw the Young diagrams rotated by 90° counterclockwise. Then we can interpret a Young diagram of a partition of length $\leq n$ as a Tetris position on a Tetris game field of width n , with non-increasing height of columns (from left to right).

Definition 1. For a partition μ of length $\leq n$, denote $T_n(\mu)$ the set of partitions, Young diagram of which can be obtained from the Young diagram of μ by either dropping to it a 1×1 block, or taking a top 1×1 block in one of the columns of the Young diagram for μ and raising it back above the top of the Tetris game field.

Note that in contrast to the L-tris, defined in [16] for the description of the tensor products of representations of $SL(n)$, we don't delete complete Tetris rows here.

Then

$$\tilde{\rho}_\mu \otimes \tilde{\rho}_1 \simeq \sum_{\lambda \in T_n(\mu)} \tilde{\rho}_\lambda. \quad (6)$$

Lemma 1.

$$\tilde{\rho}_1^{\otimes m} \simeq \sum_{\substack{\lambda \in L_n, |\lambda| \leq m \\ |\lambda| \equiv m \pmod{2}}} \tilde{f}_m^\lambda(n) \tilde{\rho}_\lambda, \quad (7)$$

where $|\lambda|$ denotes the weight (i. e. the sum of all parts) of a partition λ and $\tilde{f}_m^\lambda(n)$ is the number of symplectic lattice words in the alphabet $C_n = \{1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}\}$ of length m and weight $x_\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n}$, where a word $i_1 \dots i_m$ is called a symplectic lattice word iff the weight of each its initial subword $x_{|i_1|}^{\text{sgn } i_1} \dots x_{|i_k|}^{\text{sgn } i_k}$ equals $x^\tau = x_1^{\tau_1} \dots x_n^{\tau_n}$ for a partition τ , i. e. $\tau_1 \geq \dots \geq \tau_n \geq 0$ where $\text{sgn } i = 1$, $\text{sgn } \bar{i} = -1$, $|i| = i$, $|\bar{i}| = i$ for $i \in C_n$.

Proof. By induction on m , by iteration of (6), the Young diagrams of the partitions λ in the right hand side of (7) can be obtained by dropping or raising m 1×1 blocks on the Tetris game field as were described above. Writing each time when a block drops or raises the number of the column where it drops, or overlined number of the column from the top of which it raises, one obtains a symplectic lattice word, because the definition of the symplectic lattice word means exactly that we have a Young diagram on each step of our game. \square

Corollary 1. The dimension of the space of $Sp(2n)$ -invariants in $V^{\otimes m}$ where V is the defining representation of $Sp(2n)$, equals $\tilde{f}_m^{\vec{0}}(n)$, the number of balanced symplectic lattice words in the alphabet C_n , where balanced means that the word contains the same number of i 's and \bar{i} 's for every i from 1 to n .

Proof. Since the dimension of the space of invariants is a coefficient at $\tilde{\rho}_0$ in (7), it equals $\tilde{f}_m^0(n)$ by Lemma 1. \square

Lemma 2. *The subspace of $Sp(2n)$ -invariants of $V \otimes V$ is one-dimensional and we can choose the fundamental form ω defined in (5) as a basis element in this subspace.*

Proof. By the definition of $Sp(2n)$, ω is invariant. By Corollary 1, the dimension of the space of invariants is equal to $\tilde{f}_2^0(n) = 1$ since there is exactly one possible S-tetris game leaving nothing after two steps: drop the 1×1 block at the first column and then raise it. $1\bar{1}$ is the corresponding unique balanced symplectic lattice word of length 2. \square

Definition 2. A symplectic $2n$ -wave graph is a graph with the vertices $1, 2, \dots, m$, each connected component of which is a path of length ≥ 1 (i. e. it can't be a point), edges of which can be drawn in the book with n pages, i. e. n copies of the upper half-plane, glued along \mathbb{R} , such that the first edge of each connected component, $\{i_1 i_2\}$, is drawn on the first page; each edge $\{i_j i_{j+1}\}$ consequent to the edge $\{i_{j-1} i_j\}$ drawn on k -th page, is drawn either on $(k+1)$ -th or $(k-1)$ -th page, if they exist, I mean that the edge consequent to the edge drawn on the first page, must be drawn on the second page and the edge consequent to the edge drawn on the n -th page, must be drawn on $(n-1)$ -th page; the last edge of the path, $\{i_l i_{l+1}\}$, supposed to be drawn on the first page; we suppose also that $i_1 < i_2 < \dots < i_{l+1}$ and edges of our symplectic wave graph don't intersect.

Lemma 3. *The number of symplectic $2n$ -wave graphs with m vertices is nonzero iff m is even in which case it equals $\tilde{f}_m^0(n)$.*

Proof. We'll construct a bijection between the set of balanced symplectic lattice words of length m in the alphabet C_n and the set of symplectic $2n$ -wave graphs with m vertices. After that Lemma 3 will follow from Corollary 1.

First, construct a mapping from graphs to words. For each vertex i of a symplectic $2n$ -wave graph denote $\alpha_i = k \in C_n$ such that $|k|$ is the largest number of pages of the book containing edges ending in i and $\text{sgn } k = \text{sgn}(j - i)$ where $\{ij\}$ is the edge lying on the k -th page; in other words, $\text{sgn } k = 1$ for the initial vertex of the path and if running along the path through i we come from a page with a smaller number to a page with a larger number, or $\text{sgn } k = -1$ for the last vertex of a path and if running along the path through i we come from a page with a larger number to a page with a smaller number. Then the word $\alpha_1 \dots \alpha_m$ must be a balanced symplectic lattice word. Since

the weight of an initial subword of this word is a product of weights of initial subwords of paths, it is enough to show that the word corresponding to a path is a symplectic lattice word.

For paths, we'll use induction on their lengths. There is the unique path of length 1, the corresponding word $1\bar{1}$ is a balanced symplectic lattice word. Suppose that the words corresponding to paths of length less than $l > 1$, are balanced symplectic lattice words. Take a path of length l .

If it doesn't have other edges on its first page except the first edge and the last one, then deleting the first and the last vertices and the first page, we obtain a $2(n-1)$ -wave path of length $l-2$ (after the appropriate renaming of the pages and vertices). By supposition, the word $2\dots\bar{2}$ corresponding to it must be a balanced symplectic lattice word in the alphabet $C_n \setminus \{1, \bar{1}\}$ containing only one 2 (initial) and only one $\bar{2}$ (final). Thus, the weight of each initial subword must be of type $x_2x_3\dots x_k$ for some $k \geq 2$, or 1 if this subword is the whole word. Thus, after adding deleted vertices 1 and $\bar{1}$, the weights of the initial subwords will be $x_1x_2x_3\dots x_k$ for some k or 1 for the whole word that means that it will be a balanced symplectic lattice word.

If the path of length l contains other edges than the first and the last on its first page, let one of them be the edge $\{i(i+1)\}$, then $\alpha_i = \bar{2}$, $\alpha_{i+1} = 2$. In this case the complete subgraph of the given path with vertices from 1 to $i+1$, edges of which are drawn on the same pages, is a symplectic $2n$ -wave path and analogously for the complete subgraph with vertices from i to $l+1$. The length of these paths is less than l , thus the words corresponding to them are balanced symplectic lattice words. For $j \leq i$, the initial subword $\alpha_1\dots\alpha_j$ of the original word, is the same as for the first of two new paths, thus the weight of it has the required form. In a new word $\alpha_{i+1} = \bar{1}$. Thus the weight of a subword $\alpha_1\dots\alpha_i$ is x_1 , in both words, old and new, as well as the weight of the initial one-letter word α_i of another new word. Hence all the other weights of the initial subwords $\alpha_1\dots\alpha_j$ with $j > i$ of the original word, will be the same as for the initial subword $\alpha_i\dots\alpha_j$ of the second new word, thus they have the required form as well. By induction, we have proven than the constructed above word is a balanced symplectic lattice word.

Now we construct the inverse mapping, from words to graphs. Let $\alpha = \alpha_1\dots\alpha_m$ be a balanced symplectic lattice word. To construct all the edges on the k -th page of our book, take the letters α_i of this word such that $|\alpha_i| = k$ or $|\alpha_i| = k+1$. Write them in the increasing order of their indices and rename all the letters $\alpha_i = k+1$ to \bar{k} and all the letters $\alpha_i = \overline{k+1}$ to k . We get a letter in alphabet $\{k, \bar{k}\}$. The same as usual, the same as for outerplanar graphs [14, 15, 16, 21], draw the

outerplanar graph on k -th page of our book with the chosen vertices: one way of doing that is to put a left bracket instead of k , right bracket instead of \bar{k} and connect the corresponding left and right brackets. Doing that for all k from 1 to m , we obtain a graph which must be a symplectic $2n$ -wave graph. Indeed, it is easy to check all the requirements. Also, by construction, these two mappings are mutually inverse. Thus, we constructed a bijection between the set of balanced symplectic lattice words of length m in the alphabet C_n and the set of symplectic $2n$ -wave graphs with m vertices. \square

Lemma 4. *The number of connected symplectic $2n$ -wave graphs with m vertices equals $c_{m-2}(\mathcal{P}_n)$, the number of walks of length $(m-2)$ from 1 to 1 on the path \mathcal{P}_{n-1} of length $(n-1)$, i. e. a simple graph with n vertices $1, \dots, n$ and $(n-1)$ edges $\{i, i+1\}$ for i running from 1 to $(n-1)$.*

Proof. The same as in the proof of Lemma 3, we'll construct a bijection. Let us think that the vertices of \mathcal{P}_{n-1} are the numbers of the pages of a book in which our connected symplectic wave graph is drawn. The first edge of our graph is drawn on the first page, it means that in the beginning we are in the initial vertex 1 of \mathcal{P}_{n-1} . The second edge is drawn on the second page: it corresponds to moving from 1 to 2 in \mathcal{P}_{n-1} ; and so on, if the edge consequent to an edge drawn on k -th page, is drawn on $(k \pm 1)$ -th page, we are moving from k to $(k \pm 1)$ in \mathcal{P}_{n-1} . The last edge is drawn on the first page, it means that at the end of our walk on \mathcal{P}_{n-1} we are returning to the vertex 1. Conversely, for each walk $\alpha_1 \dots \alpha_{m-1}$ from 1 to 1 on \mathcal{P}_{n-1} , we can construct a connected symplectic $2n$ -wave graph with m vertices, drawing its k -th edge on α_k -th page. These two mappings are mutually inverse. Thus we constructed a bijection. \square

By Lemma 3, we have the same number of symplectic wave graphs as we need. Let us construct the corresponding invariants.

Definition 3. For a symplectic $2n$ -wave graph G having 2 vertices $\{1, 2\}$ and an edge between them, drawn on the first page, denote $t_G = \omega$. For a non-connected symplectic $2n$ -wave graph $G = G_1 \amalg G_2$ define

$$t_G = (t_{G_1^o} \otimes t_{G_2^o})^\sigma \tag{8}$$

where G_1^o and G_2^o are symplectic $2n$ -wave graphs obtained from G_1 and G_2 by reindexing their vertices in the same order and σ is the permutation putting the vertices of graphs G_1^o and G_2^o in their tensor product (8) on their correct positions in G . For a path G with $m > 2$ vertices with a

balanced symplectic lattice word $1\beta\bar{1}$ where $\beta = \beta_1 \dots \beta_{m-2}$ is a balanced symplectic lattice word in the alphabet $C_n \setminus \{1, \bar{1}\}$ define

$$t_G = \omega \wedge t_B \quad (9)$$

where B is $2n$ -wave graph corresponding to the word $(\beta_1 - 1) \dots (\beta_{m-2} - 1)$ in the alphabet C_n and for 2-tensor α and $(m-2)$ -tensor β we define m -tensor

$$\alpha \wedge \beta = \sum_{1 \leq i < j \leq m} (-1)^{i+j-3} (\alpha \otimes \beta)^{\sigma_{ij}} \quad (10)$$

where $\sigma_{ij} \in S_m$ is the permutation mapping 1 to i , 2 to j and other elements to the vacant places in increasing order, i. e. $\sigma_{ij}(3) < \sigma_{ij}(4) < \dots < \sigma_{ij}(m)$.

More general,

Definition 4. For a k -tensor α and $(m-k)$ -tensor β we define m -tensor

$$\alpha \wedge \beta = \sum_{1 \leq i_1 < \dots < i_k \leq m} (-1)^{(i_1-1)+\dots+(i_k-k)} (\alpha \otimes \beta)^{\sigma_{i_1 \dots i_k}} \quad (11)$$

where $\sigma_{i_1 \dots i_k} \in S_m$ is the permutation mapping 1 to i_1 , 2 to i_2 , \dots , k to i_k and other elements to the vacant places in increasing order, i. e. $\sigma_{i_1 \dots i_k}(k+1) < \dots < \sigma_{i_1 \dots i_k}(m)$.

Lemma 5. *The wedge products of tensors defined above is associative, distributive respective to the addition and satisfies*

$$\alpha \wedge \beta = (-1)^{k(m-k)} \beta \wedge \alpha \quad (12)$$

for a k -tensor α and $(m-k)$ -tensor β .

Proof. Associativity and distributivity follows directly from Definition 4. For (12), note that the sign of an item of the sum in (11) coincides with the sign of the corresponding permutation $\sigma_{i_1 \dots i_k}$ since this permutation has exactly $(i_1 - 1) + \dots + (i_k - k)$ inversions. Permutations in the left hand side and the right hand side of (12) differs on $\sigma_{i_1 \dots i_k}$ with $i_j = m - k + j$ having $k(m - k)$ inversions. \square

Lemma 6. *For any symplectic $2n$ -wave graph G , in the lexicographical order of the monomial basis of $V^{\otimes m}$ corresponding to the ordering $p_1 < p_2 < \dots < p_n < q_n < \dots < q_2 < q_1$ of B_{2n} , the monomial*

$$b_{\alpha(G)} = b_{\alpha_1} \otimes \dots \otimes b_{\alpha_m} \quad (13)$$

where $\alpha(G)$ is a balanced symplectic lattice word corresponding to the symplectic $2n$ -wave graph G and

$$b_i = \begin{cases} p_i & \text{if } \text{sgn } i = 1, \\ q_{|i|} & \text{if } \text{sgn } i = -1, \end{cases} \quad (14)$$

is the minimal monomial with a non-zero coefficient in t_G .

Proof. The proof is not very simple and we'll do it in a few steps. First, recall that

$$\omega^{\wedge k} = \underbrace{\omega \wedge \cdots \wedge \omega}_{k \text{ times}} = k! \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq n \\ \sigma \in S_{2k}}} (-1)^{\text{inv } \sigma} (p_{i_1} \otimes \cdots \otimes p_{i_k} \otimes q_{i_k} \otimes \cdots \otimes q_{i_1})^\sigma. \quad (15)$$

For a connected symplected $2n$ -wave graph corresponding to a walk $12 \dots (k-1)k(k-1) \dots 21$ according to the bijection constructed in Lemma 4, the invariant t_G is $\omega \wedge \cdots \wedge \omega$. It follows from (15) that the basis monomial corresponding to the word $12 \dots k\bar{k} \dots \bar{2}1$ is the minimal monomial with a nonzero coefficient for that case.

By induction we can deduce that for other connected symplectic $2n$ -wave graphs

$$t_G = \omega^{\wedge k} \wedge (t_{G_1} \otimes t_{G_2}) \quad (16)$$

for some k and symplectic $2n$ -wave graphs G_1 and G_2 .

Using (16), we can prove by induction that if we write down all the letters i and \bar{i} , in the same order, from the index word of a monomial with a non-zero coefficient in t_G , we get a word in the 2-letter alphabet $\{i, \bar{i}\}$ such that the weight of each initial subword is either x or x^{-1} ; for any i from 1 to n . Note that in a word $\alpha_1 \dots \alpha_{2l}$ in 2-letter alphabet ± 1 with this condition, the letters α_{2k-1} and α_{2k} have different signs for all k from 1 to l .

Denote $M_m(2n)$ the set of balanced words of length m in the alphabet C_n satisfying the condition above, i. e. such that for every i from 1 to m the word in 2-letter alphabet $\{i, \bar{i}\}$ containing all the entries of i and \bar{i} , in the same order, has the weight either x or x^{-1} for each initial subword. Note that if we have

$$\alpha_{2k-1} = \bar{i}, \quad \alpha_{2k} = i \quad (17)$$

in one of such words, then transposing these letters \bar{i} and i in the original word we obtain a word in $M_m(2n)$ less than original one in the lexicographical order of the words mentioned in the statement of the Lemma.

Denote $M_m^+(2n)$ the subset of $M_m(2n)$ containing such words that for every i from 1 to m the word in 2-letter alphabet $\{i, \bar{i}\}$ containing all the entries of i and \bar{i} , in the same order, is $\bar{i}\bar{i}\dots i\bar{i}$, i. e. with the first letter i and alternating of the letters on every step.

Define for every word $\alpha = (\alpha_1 \dots \alpha_m) \in M_m^+(2n)$ its *pattern* as a word

$$\text{pat}(\alpha) = \text{sgn } \alpha_1 \dots \text{sgn } \alpha_m \quad (18)$$

of length m in the alphabet ± 1 . By definition of $M_m^+(2n)$, the sum of all entries of $\text{pat}(\alpha)$ is 0 and each initial subword of this word has a nonnegative sum of its entries.

Conversely, for every word $\delta = \delta_1 \dots \delta_m$ of an even length m in the alphabeth ± 1 with zero sum of entries, each initial subword of which has a nonnegative sum of entries, we define its lattice word

$$\text{lat}(\delta) = \alpha_1 \dots \alpha_m \in M_m^+(2n) \quad (19)$$

assuming $\alpha_1 = 1$, $\alpha_m = -1$ and for other k from 1 to m

$$|\alpha_k| = \max\{s_k(\delta), s_{k-1}(\delta)\}, \quad \text{sgn } \alpha_k = \delta_k \quad (20)$$

where s_k means the sum of the first k entries.

By definition, the pattern of the word $\text{lat}(\delta)$ is δ . By induction on k , reading the word from the beginning to the end, we can check that $\text{lat}(\delta)$ is the smallest word in $M_m^+(2n)$ of the pattern δ with respect to the lexicographical order. Indeed, the smallest possible first letter is 1. If $\delta_2 = 1$, we can't have the second letter 1 again, because the entries of 1 and $\bar{1}$ must alternate; thus the smallest second letter is 2. Otherwise, if $\delta_2 = -1$, the only possibility for the second letter is $\bar{1}$. Later, if we have $s_{k-1} = i$, $s_k = i + 1$, we can't use $1, 2, \dots, i$ for the k -th letter, because they were used earlier once more than the corresponding overline numbers (each of them), thus the smallest possible choice is $(i + 1)$. If we have $s_{k-1} = i$, $s_k = i - 1$, we must use one of $\bar{1}, \dots, \bar{i}$ as the k -th letter and the smallest possible choice is \bar{i} . Thus, $\text{lat}(\delta)$ is the smallest word in $M_m^+(2n)$ of the pattern δ .

Introduce the inverse lexicographical order on the patterns, i. e. lexicographical order corresponding to the ordering $1 \prec -1$. It is easy to see that for the patterns $\epsilon \prec \delta$ we have $\text{lat}(\epsilon) \prec \text{lat}(\delta)$. Indeed, if k is the smallest integer such that $\epsilon_k \prec \delta_k$, i. e. $\epsilon_k = 1$, $\delta_k = -1$, then by construction the words $\text{lat}(\epsilon)$ and $\text{lat}(\delta)$ have the same letters on the first $k - 1$ places and

$$(\text{lat}(\epsilon))_k < (\text{lat}(\delta))_k \quad (21)$$

since

$$(\text{lat}(\epsilon))_k \in \{1, 2, \dots, n\}, \quad (\text{lat}(\delta))_k \in \{\bar{1}, \bar{2}, \dots, \bar{n}\}. \quad (22)$$

Now look at the patterns of the indices of the monomials with non-zero coefficients in t_G for a connected symplectic $2n$ -wave graph G . For any pattern of the analogous monomials for the symplectic $2n$ -wave graph B defined in Definition 3, the smallest pattern that it can give us for t_G is not less than if we add 1 in the beginning of it and -1 at the end. Indeed, if it has i 1's at the beginning before the first -1 , adding 1 at the beginning gives $(i+1)$ 1's at the beginning; the same as adding 1 before the first -1 , and it gives a smaller pattern than one with i 1's in the beginning obtained by adding 1 after the first -1 . Analogously, adding -1 before the last 1 gives a larger word than the adding -1 at the end since the line of -1 's containing it becomes longer. The smallest pattern that we can obtain in this way is when we add 1 at the beginning and -1 at the end to the smallest pattern for B .

The smallest pattern for B with the corresponding lattice word $1\bar{1}$ is $1-1$ since $t_B = \omega$ in that case. We have $\text{lat}(1-1) = 1\bar{1}$. Prove by induction on m that the same is true in general, i. e. the smallest pattern of the basis monomials of t_G with non-zero coefficients is $\text{pat}(\alpha(G))$ where $\alpha(G)$ is the corresponding lattice word for a symplectic $2n$ -wave graph G . Since for non-connected graphs we obtain the smallest pattern by combining the patterns of the connected components, it is enough to prove that for a connected G supposing by inductive hypothesis that the smallest pattern for B is $\text{pat}(\alpha(B))$. Since we can obtain $\text{pat}(\alpha(G))$ from $\text{pat}(\alpha(B))$ by adding 1 at the beginning and -1 at the end, this is true as we had shown in the previous paragraph.

Now when we know that the smallest possible pattern is $\text{pat} \alpha(G)$ and the smallest lattice word with this pattern is

$$\text{lat}(\text{pat}(\alpha(G))) = \alpha(G), \quad (23)$$

the only thing that we have to do on the last step of our proof of Lemma 6, is to check that the coefficient at $b_{\alpha(G)}$ is non-zero. By induction, we'll prove that this coefficient is positive.

First, reading the word $\alpha(G)$ for a connected symplected $2n$ -wave graph G , we see that all the odd digits $1, 3, \dots$ are located on odd places, all even digits $2, 4, \dots$ are on even places, all the overlined odd digits are on even places and all the overlined even digits are on odd places. Indeed, it is true for the first 1, and the next letter after k can be either $k+1$, or k ; the next letter after \bar{k}

can be either $\overline{k-1}$ or k , i. e. the parity changes on every step according to our hypothesis. Thus, by induction, it is true.

Using that, we can prove, again by induction, that for a connected $2n$ -wave graph G all the monomials from $M_m^+(2n)$ have non-negative coefficients. The same is true for B as well since its connected components don't interlace. Thus all the items giving a monomial $b_{\alpha(G)}$ in the wedge product (9) have non-negative coefficients, it means that they can't cancel and at least one of them, obtained from $p_1 \wedge q_1 \wedge t_B$ by putting p_1 in the first place, q_1 in the last place and $b_{\alpha_+(B)}$ between them, where $\alpha_+(B)$ is a word obtained from $\alpha(B)$ by changing k to $k+1$ and \overline{k} to $\overline{k+1}$ for all k from 1 to $(n-1)$, has a positive coefficient since the coefficient at $b_{\alpha(B)}$ in t_B is non-zero by induction hypothesis, and $b_{\alpha_+(B)}$ has the same coefficient because t_B is $Sp(2n)$ -invariant and the linear transformation

$$p_k \mapsto p_{k+1}, \quad q_k \mapsto q_{k+1} \quad (24)$$

for k from 1 to $n-1$ and

$$p_n \mapsto p_1, \quad q_m \mapsto q_1, \quad (25)$$

is symplectic, therefore preserve t_B . □

Theorem 1. *Tensors t_G parametrized by all $2n$ -wave graphs with m vertices, form a basis in the space of $Sp(2n)$ -invariants in $V^{\otimes m}$.*

Proof. Lemma 3 and Corollary 1 show that the number of symplectic $2n$ -wave graphs with m vertices is exactly the same as the dimension of the corresponding space of $Sp(2n)$ -invariants. By Definition 3 and Lemma 2, since tensor product of invariants are invariant as well as the result of permutation of tensor factors, for any symplectic $2n$ -wave graph G , tensor t_G is $Sp(2n)$ -invariant. Hence if we prove linear independence of the set of t_G , our theorem will be proven. The proof is completely analogous to the proof of the particular case $n=2$ and the analogous theorem for $SL(n)$ given in my articles [14, 15, 16].

Denote B the standard basis of $V^{\otimes m}$, consisting of $(2n)^m$ tensor products $X_1 \otimes \cdots \otimes X_m$ with $X_1, \dots, X_m \in \{p_1, \dots, p_n, q_1, \dots, q_n\}$. Suppose that B is ordered lexicographically corresponding to the ordering $p_1 < p_2 < \cdots < p_n < q_n < \cdots < q_2 < q_1$. By Lemma 6, $b_{\alpha(G)} \in B$ is the minimal

element of B with a non-zero coefficient in the decomposition of t_G . Note that for different graphs G the lattice words $\alpha(G)$ are different. So, we have $\tilde{f}_m^0(n)$ elements $b_{\alpha(G)}$ —one for each G .

To prove the linear independence of the set of t_G , we can show that the rank of the $\tilde{f}_m^0(n) \times (2n)^m$ matrix of the coefficients of t_G in the basis B , is equal to $\tilde{f}_m^0(n)$. To do that, we can find a non-zero $\tilde{f}_m^0(n) \times \tilde{f}_m^0(n)$ minor of that matrix. Consider the $\tilde{f}_m^0(n) \times \tilde{f}_m^0(n)$ submatrix with rows numbered by G ordered the same way as $b_{\alpha(G)}$ and columns corresponding to $b_{\alpha(G)}$. As we noticed above, $b_{\alpha(G)}$ is the first element with a nonzero coefficient in the row G and this coefficient is non-zero by Lemma 6. So, this matrix is upper triangular with non-zero elements on the diagonal, therefore its determinant is not 0, that completes the proof of the linear independence of t_G . \square

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